

# Doubly covariant action principle of singular hypersurfaces in general relativity and scalar-tensor theories

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An action principle of singular hypersurfaces in general relativity and scalar-tensor type theories of gravity in the Einstein frame is presented without assuming any symmetry. The action principle is manifestly doubly covariant in the sense that coordinate systems on and off a hypersurface are disentangled and can be independently specified. It is shown that, including variation of the metric, the position of the hypersurface and matter fields, the variational principle gives the correct set of equations of motion: the Einstein equation off the hypersurface, Israel's junction condition in a doubly covariant form and equations of motion of matter fields including the scalar fields. The position of the hypersurface measured from one side of the hypersurface and that measured from another side can be independently varied as required by the double covariance.

## I. INTRODUCTION

Spatially extended objects such as branes, membranes, shells and walls have been playing many important roles in recent progress in theoretical physics including string theory [1], particle phenomenology [2–4], theory of black holes [5–7], cosmology [8,9] and so on. Hence, it is important to investigate dynamics of such extended objects. In particular, the so called brane world scenario is based on the idea that our four-dimensional universe may be a world volume of a brane in a higher dimensional spacetime [2–4,10]. Thus, in the brane world scenario the dynamics of the brane is the dynamics of our universe itself and is of the most physical importance.

It is well-known and is the most commonly adopted picture that the dynamics of extended objects is elegantly described as geometrical imbedding of world-volume surfaces into spacetime in a certain limit. In particular, in the case of codimension 1, or when the world-volume surface is a hypersurface, the geometrical description becomes simpler than other cases with higher codimension. Actually, in general relativity or other theories of gravity in the Einstein frame, the classical dynamics of a hypersurface is perfectly described by Israel's junction condition [11].

One of the main advantages of the junction condition is that it is manifestly doubly covariant in the sense that coordinate systems on and off a hypersurface are disentangled and can be independently specified. More precisely, there are three independent coordinate systems: that on the hypersurface, those in two regions separated by the hypersurface. In the brane world point of view, the double covariance is important since it allows us to separate the coordinate system in our world from that in the higher dimensional spacetime.

Once the classical dynamics is understood, one would usually like to understand quantum mechanical dynamics [12–20]. For this purpose, we would like to obtain the action principle for the system including a hypersurface.

The easiest way to obtain the action may be to adopt the Gaussian normal coordinate system based on the hypersurface and to consider the Einstein-Hilbert action with a delta function source. The action obtained in this way gives the correct set of equations in the coordinate system, provided that the position of the hypersurface and coordinates in a neighborhood of the hypersurface are fixed by the Gaussian normal coordinate condition. However, in this method we lose the double covariance: coordinates on the hypersurface is a part of coordinates off the hypersurface so that the coordinates satisfy the Gaussian normal coordinate condition. The loss of the double covariance is regrettable.

Actually, as far as the author knows, a doubly covariant action principle has not yet been obtained in the literature. One of the main difficulties seems due to the fact that the spacetime metric on one side of the hypersurface and that on another side are independent variables in the variational principle. Hence, a question arises: How can we ensure the regularity of the intrinsic geometry of the hypersurface without entangling the coordinate systems on and off the hypersurface? This question will be answered in this paper as a manifestly covariant action principle will be presented.

Another difficulty is due to the fact that the double covariance requires inclusion of the position of the hypersurface as a dynamical variable in the action principle. Actually, in the doubly covariant formulation of the junction condition, it is easy to see that variables specifying the position are not invariant under coordinate transformation and should not be fixed [21]. More about why we need to include the position of the hypersurface will be explained from the brane world point of view in Sec. V. Here, we mention that, since coordinate systems in two sides of the hypersurface are independent, the position of the hypersurface measured from one side and that measured from another side should be independently varied in the variational principle.

It may be worth while reviewing the present status in the literature regarding the second difficulty. However, the author knows only a few papers referring this point. Here, we only quote a sentence from one of them: ‘*The variational equations that arose from the unreduced Hamiltonian action were not strictly consistent in a distributional sense, but we were able to localize the ambiguity into the single equation that arises by varying the action with respect to the shell position* [19]’. One might think that another paper [20] had obtained the correct set of equations, but in that paper the position of the hypersurface measured from one side and that measured from another side can not be varied independently. Actually, if we would simply variate the position of the hypersurface measured from one side and that measured from another side independently, then the variational principle presented in ref. [20] would give wrong equations. Moreover, in both of these papers, the hypersurface represents only a dust shell and the coordinate systems on and off the hypersurface are not independent. One of them [19] assumes spherical symmetry, too.

The purpose of this paper is to provide a manifestly doubly covariant action principle of singular hypersurfaces in general relativity and scalar-tensor type theories of gravity in the Einstein frame without assuming any symmetry. Besides the scalar fields included in the scalar-tensor type theories, any kind of matter Lagrangian density on the hypersurface, which may depend also on the pullback of the scalar fields, can be included. It is shown that, including variation of the metric, the position of the hypersurface and matter fields, the variational principle gives the correct set of equations of motion: the Einstein equation off the hypersurface, Israel’s junction condition in a doubly covariant form and equations of motion of matter fields including the scalar fields. As required by the double covariance, the position of the hypersurface measured from one side of the hypersurface and that measured from another side can be independently varied.

This paper is organized as follows. In Sec. II a doubly covariant action of a singular hypersurface is derived from the standard Einstein-Hilbert action. In Sec. III the variation of the action is calculated for the variations of the metric and the position of the hypersurface, and the corresponding equations are obtained. In Sec. IV the variation of the action corresponding to the variations of scalar fields are evaluated. Sec. V is devoted to a summary of this paper and some discussions.

## II. ACTION OF SINGULAR HYPERSURFACE

Let us consider a  $D$ -dimensional spacetime  $(\mathcal{M}, g_{MN})$  and a timelike or spacelike hypersurface  $\Sigma$  which separates  $\mathcal{M}$  into two regions,  $\mathcal{M}_+$  and  $\mathcal{M}_-$ . Since we would like to consider  $\Sigma$  as a physical object (eg. the world-volume of a brane or the world-volume of a bubble wall in a first-order phase transition) or a physical event (eg. instantaneous global phase transition [22]), we assume that the  $(D-1)$ -dimensional intrinsic geometry on  $\Sigma$  is regular. On the other hand, the  $D$ -dimensional geometry is not necessarily regular on  $\Sigma$ .

In the following arguments we shall estimate the action for the system including the singular hypersurface  $\Sigma$ . We assume that the system is described by the action <sup>1</sup>

$$I_{tot} = I_{EH} + I_{matter}, \quad (1)$$

where  $I_{EH}$  is the Einstein-Hilbert action with a cosmological constant

$$I_{EH} = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^D x \sqrt{|g|} (R - 2\Lambda), \quad (2)$$

and  $I_{matter}$  is the matter action of the form

$$I_{matter} = \int_{\mathcal{M}_+} d^D x_+ \mathcal{L}_+ + \int_{\mathcal{M}_-} d^D x_- \mathcal{L}_- + \int_{\Sigma} d^{D-1} y \mathcal{L}_0. \quad (3)$$

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<sup>1</sup>For simplicity we do not consider the boundary of  $\mathcal{M}$ , but it is easy to take it into account by imposing suitable boundary conditions and introducing boundary terms appropriate for the boundary condition.

Here,  $\{x_{\pm}^M\}$  are  $D$ -dimensional coordinate systems in  $\mathcal{M}_{\pm}$ , respectively, and  $\{y^{\mu}\}$  is a  $(D-1)$ -dimensional coordinate system in  $\Sigma$ . The three coordinate systems can be independent from each other.

In order to evaluate the gravitational part of the action, we first regularize the  $D$ -dimensional geometry in a neighborhood of  $\Sigma$  by introducing the finite thickness  $\delta$  of the object corresponding to  $\Sigma$ . Of course, in the final step below, we shall take the limit  $\delta \rightarrow +0$ , where the hypersurface becomes singular again. Namely, we consider the prescription

$$\begin{aligned} I_{EH} &= \lim_{\delta \rightarrow +0} (I_+^{\delta} + I_0^{\delta} + I_-^{\delta}), \\ I_0^{\delta} &= \frac{1}{2\kappa^2} \left[ \int_{\mathcal{M}_0^{\delta}} d^D x \sqrt{|g|} (R - 2\Lambda) + 2\epsilon \int_{\mathcal{B}_+^{\delta}} d^{D-1} y \sqrt{|q|} K - 2\epsilon \int_{\mathcal{B}_-^{\delta}} d^{D-1} y \sqrt{|q|} K \right], \\ I_{\pm}^{\delta} &= \frac{1}{2\kappa^2} \left[ \int_{\mathcal{M}_{\pm}^{\delta}} d^D x \sqrt{|g|} (R - 2\Lambda) \mp 2\epsilon \int_{\mathcal{B}_{\pm}^{\delta}} d^{D-1} y \sqrt{|q|} K \right], \end{aligned} \quad (4)$$

where  $\mathcal{M}_0^{\delta}$  is a spacetime neighborhood of  $\Sigma$  representing the regularized object,  $\mathcal{M}_{\pm}^{\delta}$  are the two regions separated by  $\mathcal{M}_0^{\delta}$  so that

$$\mathcal{M}_0^{\delta} \supset \Sigma, \quad \mathcal{M}_{\pm}^{\delta} \subset \mathcal{M}_{\pm}, \quad \lim_{\delta \rightarrow +0} \mathcal{M}_{\pm}^{\delta} = \mathcal{M}_{\pm}, \quad (5)$$

and  $\mathcal{B}_{\pm}^{\delta}$  is the boundary between  $\mathcal{M}_0^{\delta}$  and  $\mathcal{M}_{\pm}^{\delta}$ , respectively. Note that surface terms have been included in  $I_{0,\pm}^{\delta}$  for later convenience but that these exactly cancel each other on common boundaries  $\mathcal{B}_{\pm}^{\delta}$ . Each surface term is defined as an integral over the  $(D-1)$ -dimensional intrinsic coordinates  $y^{\mu}$  on  $\mathcal{B}_{\pm}^{\delta}$ ,  $q$  is the determinant of the induced metric,  $K$  is the trace of the extrinsic curvature associated with the unit normal  $n^M$  directed from  $\mathcal{M}_0^{\delta}$  to  $\mathcal{M}_+^{\delta}$  or from  $\mathcal{M}_0^{\delta}$  to  $\mathcal{M}_-^{\delta}$ , and  $\epsilon = g_{MN} n^M n^N = \pm 1$ .

Next, in order to estimate  $I_0^{\delta}$ , we foliate  $\mathcal{M}_0^{\delta}$  by such a one-parameter family of hypersurfaces  $\Sigma_{\tau}$  that  $\Sigma_0$  coincides with  $\Sigma$  and that  $\Sigma_{\pm 1}$  coincides with the boundary  $\mathcal{B}_{\pm}^{\delta}$ , respectively. Hence, we can decompose the  $D$ -dimensional Ricci scalar  $R$  as

$$R = R^{(D-1)} + \epsilon K^2 - \epsilon K^{\mu\nu} K_{\mu\nu} - 2\epsilon (K n^M - n_{;N}^M n^N)_{;M}, \quad (6)$$

where  $R^{(D-1)}$  is the Ricci scalar of the  $(D-1)$ -dimensional induced metric on  $\Sigma_{\tau}$ , the semicolon represents the covariant derivative compatible with  $g_{MN}$ ,  $n^M$  is the unit normal to  $\Sigma_{\tau}$  directed towards  $\mathcal{B}_+^{\delta}$ ,  $\epsilon = g_{MN} n^M n^N = \pm 1$ ,  $K_{\mu\nu}$  is the extrinsic curvature associated with  $n^M$ , the indices  $\{\mu, \nu\}$  are raised by the inverse of the induced metric, and  $K = K_{\mu}^{\mu}$ . By integrating over  $\mathcal{M}_0^{\delta}$  and taking the limit  $\delta \rightarrow +0$ , we obtain

$$I_0^{\delta} = \frac{1}{2\kappa^2} \int_{\mathcal{M}_0^{\delta}} d^D x \sqrt{|g|} \left( R^{(D-1)} + \epsilon K^2 - \epsilon K^{\mu\nu} K_{\mu\nu} - 2\Lambda \right) \rightarrow 0 \quad (\delta \rightarrow +0). \quad (7)$$

Here, we have used the assumption that the intrinsic geometry on  $\Sigma$  is regular even in the limit  $\delta \rightarrow +0$ . We have also assumed that the extrinsic curvature remains finite.

Therefore, we obtain the following form of the Einstein-Hilbert action for the system including the singular hypersurface  $\Sigma$ .

$$\begin{aligned} I_{EH} &= \frac{1}{2\kappa^2} \left[ \int_{\mathcal{M}_+} d^D x_+ \sqrt{|g_+|} (R_+ - 2\Lambda_+) + \int_{\mathcal{M}_-} d^D x_- \sqrt{|g_-|} (R_- - 2\Lambda_-) \right. \\ &\quad \left. - 2\epsilon \int_{\Sigma} d^{D-1} y \left( \sqrt{|q_+|} K_+ - \sqrt{|q_-|} K_- \right) + \epsilon \int_{\Sigma} d^{D-1} y \lambda^{\mu\nu} (q_{+\mu\nu} - q_{-\mu\nu}) \right], \end{aligned} \quad (8)$$

where  $q_{\pm\mu\nu}$  is the induced metric,  $q_{\pm}$  is the determinant of  $q_{\pm\mu\nu}$ ,  $K_{\pm} = q_{\pm}^{\mu\nu} K_{\pm\mu\nu}$  is the trace of the extrinsic curvature  $K_{\pm\mu\nu}$  and  $q_{\pm}^{\mu\nu}$  is the inverse of  $q_{\pm\mu\nu}$ . In the expression (8) we have distinguished geometrical quantities in  $\mathcal{M}_+$  and  $\mathcal{M}_-$  by introducing the subscript  $\pm$ , and have allowed the cosmological constant to have different values in these two regions. We have introduced the Lagrange multiplier field  $\lambda^{\mu\nu}(y)$  to ensure the regularity of the intrinsic geometry of  $\Sigma$ . When we regularized the system and decomposed  $I_{EH}$  into  $I_0^{\delta}$  and  $I_{\pm}^{\delta}$  as in (4), we implicitly assumed that the induced metric and the extrinsic curvature are continuous across the boundaries  $\mathcal{B}_{\pm}^{\delta}$ . After taking the limit  $\delta \rightarrow +0$ , the extrinsic curvature remains finite but can be discontinuous across  $\Sigma$ . On the other hand, the induced metric should

remain continuous across  $\Sigma$  even after taking the limit  $\delta \rightarrow +0$  because of the finiteness of the extrinsic curvature. Provided that the hypersurface  $\Sigma$  is specified as the boundary of  $\mathcal{M}_\pm$  by the parametric equation

$$x_\pm^M = Z_\pm^M(y^\mu), \quad (9)$$

the induced metric and the extrinsic curvature are given by

$$\begin{aligned} q_{\pm\mu\nu}(y) &= e_{\pm\mu}^M(y)e_{\pm\nu}^N(y)g_{\pm MN}|_{x_\pm=Z_\pm(y)}, \\ K_{\pm\mu\nu}(y) &= \frac{1}{2} e_{\pm\mu}^M(y)e_{\pm\nu}^N(y)\mathcal{L}_{n_\pm}g_{\pm MN}|_{x_\pm=Z_\pm(y)}, \end{aligned} \quad (10)$$

where  $e_{\pm\mu}^M$  are vectors tangent to  $\Sigma$  defined by

$$e_{\pm\mu}^M(y) = \frac{\partial Z_\pm^M}{\partial y^\mu}, \quad (11)$$

and  $n_\pm^M$  is the unit normal to  $\Sigma$  directed from  $\mathcal{M}_-$  to  $\mathcal{M}_+$ . To be precise,  $n_+^M$  is the inward-directed unit normal to  $\Sigma$  as the boundary of  $\mathcal{M}_+$  and  $n_-^M$  is the outward-directed unit normal to  $\Sigma$  as the boundary of  $\mathcal{M}_-$ .

Finally, the total action of the system is given by (1), where  $I_{EH}$  and  $I_{matter}$  are given by (8) and (3), respectively.

### III. VARIATIONAL PRINCIPLE

In this section we derive equations of motion from the variational principle based on the action obtained in the previous section. Namely, we shall extremize the action  $I_{tot}$  with respect to the variation

$$\begin{aligned} g_{\pm MN}(x) &\rightarrow g_{\pm MN}(x) + \delta g_{\pm MN}(x), \\ Z_\pm^M(y) &\rightarrow Z_\pm^M(y) + \delta Z_\pm^M(y). \end{aligned} \quad (12)$$

In the following we omit the subscript  $\pm$  unless there is possibility of confusion.

First, it is easy to show that the integrand of the volume term in  $I_{EH}$  changes as follows.

$$\sqrt{|g|}(R - 2\Lambda) \rightarrow \sqrt{|g|}[(R - 2\Lambda) - (G^{MN} + \Lambda g^{MN})\delta g_{MN} + (\delta g^{MN}_{;N} - \delta g^{;M}_M) + O(\delta^2)], \quad (13)$$

where the semicolon represents the covariant derivative compatible with the background metric  $g_{MN}$  (not with the perturbed metric  $g_{MN} + \delta g_{MN}$ ), the indices  $M, N, \dots$  are lowered and raised by the background metric  $g_{MN}$  and its inverse  $g^{MN}$ , and  $\delta g$  is defined by  $\delta g = \delta g^M_M$ . Hence,

$$\begin{aligned} \delta \int_{\mathcal{M}_\pm} d^D x_\pm \sqrt{|g|}(R - 2\Lambda) &= - \int_{\mathcal{M}_\pm} d^D x_\pm \sqrt{|g|}(G^{MN} + \Lambda g^{MN})\delta g_{MN} \\ &\quad \mp \epsilon \int_{\Sigma} d^{D-1} y \sqrt{|q|} n^M (\delta g^N_{M;N} - \delta g^{;M}_M)|_{x_\pm=Z_\pm} \\ &\quad \mp \epsilon \int_{\Sigma} d^{D-1} y \sqrt{|q|} n_M \delta Z_\pm^M (R - 2\Lambda)|_{x_\pm=Z_\pm} \end{aligned} \quad (14)$$

The second term in the right hand side came from the total derivative in (13) and the last term is due to the change of the region to be integrated over.

Next, let us consider the surface term in  $I_{EH}$ . As shown in ref. [21] the variations of the induced metric and the extrinsic curvature are given by

$$\begin{aligned} \delta q_{\mu\nu} &= e_\mu^M e_\nu^N (\delta g_{MN} + \delta Z_{M;N} + \delta Z_{N;M}), \\ \delta K_{\mu\nu} &= \frac{\epsilon}{2} n^M n^N (\delta g_{MN} + 2\delta Z_{M;N}) K_{\mu\nu} \\ &\quad - \frac{1}{2} n^L e_\mu^M e_\nu^N \left[ 2\delta \Gamma_{LMN} + \delta Z_{L;MN} + \delta Z_{L;NM} + (R_{L'MLN} + R_{L'NLM})\delta Z^{L'} \right], \end{aligned} \quad (15)$$

where the right hand side is evaluated at  $x_\pm^M = Z_\pm^M(y)$  and

$$\delta\Gamma_{LMN} = \frac{1}{2}(\delta g_{LM;N} + \delta g_{LN;M} - \delta g_{MN;L}). \quad (16)$$

In order to make the covariant derivatives of  $\delta Z^M$  well-defined, we have to extend  $\delta Z^M$  off  $\Sigma$ . The expressions (15) are independent of the method of the extension. For details, see ref. [21]. Hence,

$$\sqrt{|q|}K \rightarrow \sqrt{|q|}K + \delta(\sqrt{|q|}K) + O(\delta^2), \quad (17)$$

where

$$\begin{aligned} \delta(\sqrt{|q|}K)/\sqrt{|q|} = & - \left( K^{\mu\nu} - \frac{1}{2}Kq^{\mu\nu} \right) \delta q_{\mu\nu} + \frac{\epsilon}{2}n^M n^N (\delta g_{MN} + 2\delta Z_{M;N})K \\ & - n^L q^{\mu\nu} e_\mu^M e_\nu^N (\delta\Gamma_{LMN} + \delta Z_{L;MN} + R_{L'MLN}\delta Z^{L'}). \end{aligned} \quad (18)$$

Combining this with the second term in (14), we obtain

$$\begin{aligned} n^M (\delta g_{M;N}^N - \delta g_{;M}) + 2\delta(\sqrt{|q|}K)/\sqrt{|q|} = & -(K^{\mu\nu} - Kq^{\mu\nu})\delta q_{\mu\nu} - 2n^M R_{MN}\delta Z^N \\ & - \frac{1}{\sqrt{|q|}} \left[ \sqrt{|q|}q^{\mu\nu} n^M e_\nu^N (\delta g_{MN} + 2\delta Z_{M;N}) \right]_{,L} e_\mu^L \end{aligned} \quad (19)$$

where we have used the equations

$$\begin{aligned} [e_\mu, e_\nu]^M &= 0, \\ e_\mu^M (e_\nu^N n_N)_{;M} &= e_\mu^M (n^N n_N)_{;M} = 0, \\ q^{\mu\nu} e_\mu^M e_\nu^N + \epsilon n^M n^N &= g^{MN}. \end{aligned} \quad (20)$$

Thus, the variation of the Einstein-Hilbert action  $I_{EH}$  is given by

$$\begin{aligned} 2\kappa^2 \delta I_{EH} = & - \int_{\mathcal{M}_+} d^D x_+ \sqrt{|g_+|} (G_+^{MN} + \Lambda_+ g_+^{MN}) \delta g_{+MN} - \int_{\mathcal{M}_-} d^D x_- \sqrt{|g_-|} (G_-^{MN} + \Lambda_- g_-^{MN}) \delta g_{-MN} \\ & + \epsilon \int_{\Sigma} d^{D-1} y \left\{ \left[ \sqrt{|q_+|} (K_+^{\mu\nu} - K_+ q_+^{\mu\nu}) + \lambda^{\mu\nu} \right] \delta q_{+\mu\nu} - \left[ \sqrt{|q_-|} (K_-^{\mu\nu} - K_- q_-^{\mu\nu}) + \lambda^{\mu\nu} \right] \delta q_{-\mu\nu} \right. \\ & + 2n_+^M (G_{+MN} + \Lambda_+ g_{+MN})|_{x_+=Z_+} \delta Z_+^N - 2n_-^M (G_{-MN} + \Lambda_- g_{-MN})|_{x_-=Z_-} \delta Z_-^N \\ & \left. + (q_{+\mu\nu} - q_{-\mu\nu}) \delta \lambda^{\mu\nu} \right\}, \end{aligned} \quad (21)$$

Now let us consider the variation of  $I_{matter}$ .

$$\begin{aligned} 2\delta I_{matter} = & \int_{\mathcal{M}_+} d^D x_+ \sqrt{|g_+|} T_+^{MN} \delta g_{+MN} + \int_{\mathcal{M}_-} d^D x_- \sqrt{|g_-|} T_-^{MN} \delta g_{-MN} \\ & + \epsilon \int_{\Sigma} d^{D-1} y \left[ \sqrt{|q|} S^{\mu\nu} \delta q_{\mu\nu} + 2F_{+M} \delta Z_+^M + 2F_{-M} \delta Z_-^M \right], \end{aligned} \quad (22)$$

where  $q_{\mu\nu}$  is either  $q_{+\mu\nu}$  or  $q_{-\mu\nu}$ , and

$$\begin{aligned} \sqrt{|g_\pm|} T_\pm^{MN}(x_\pm) &= 2 \frac{\delta}{\delta g_{\pm MN}(x_\pm)} \int_{\mathcal{M}_\pm} d^D x'_\pm \mathcal{L}_\pm, \\ \sqrt{|q|} S^{\mu\nu}(y) &= 2\epsilon \frac{\delta}{\delta q_{\mu\nu}(y)} \int_{\Sigma} d^{D-1} y' \mathcal{L}_0 \Big|_{\delta Z^M=0}, \\ \sqrt{|q|} F_{\pm M}(y) &= \mp n_{\pm M} \mathcal{L}_\pm|_{x_\pm=Z_\pm(y)} + \epsilon \frac{\delta}{\delta Z_\pm^M(y)} \int_{\Sigma} d^{D-1} y' \mathcal{L}_0 \Big|_{\delta q_{\mu\nu}=0}. \end{aligned} \quad (23)$$

Therefore,  $\delta I_{tot} = 0$  is equivalent to the following set of equations.

$$\begin{aligned} G_\pm^{MN} + \Lambda_\pm g_\pm^{MN} &= \kappa^2 T_\pm^{MN}, \\ q_{+\mu\nu} - q_{-\mu\nu} &= 0, \\ K_+^{\mu\nu} - K_-^{\mu\nu} &= \kappa^2 \left( S^{\mu\nu} - \frac{1}{D-2} S q^{\mu\nu} \right), \end{aligned} \quad (24)$$

$$F_{\pm N} = \mp n_{\pm}^M T_{\pm MN}|_{x_{\pm}=Z_{\pm}}, \quad (25)$$

and

$$\lambda^{\mu\nu} = -\sqrt{|q|}(K_{\mp}^{\mu\nu} - K_{\mp} q^{\mu\nu}). \quad (26)$$

In the right hand side of the last equation, the subscript  $-$  (or  $+$ ) should be taken when  $\mathcal{L}_0$  is written in terms of  $q_{+\mu\nu}$  (or  $q_{-\mu\nu}$ , respectively).

Note that the equations (24) are the Einstein equation and Israel's junction condition [11]. The last equation is just to determine the Lagrange multiplier field  $\lambda^{\mu\nu}$ . Although the equation (25) looks like a new independent equation, it will be shown in the next section for simple examples that the equation is compatible with equations of motion of matter fields. Therefore, the action principle gives the correct set of equations: the Einstein equation, Israel's junction condition and equations of motion of matter fields.

#### IV. SIMPLE EXAMPLES

In this section we show that for simple examples, the equation (25) is compatible with equations of motion of matter fields. The first trivial example is the case in which all matter fields are confined on the hypersurface  $\Sigma$ . This case includes a shell with an arbitrary equation of state in a vacuum and the brane world scenario in a purely gravitational bulk with a bulk cosmological constant and arbitrary matter fields on the brane. In this case, the consistency condition (25) is trivially satisfied since  $\mathcal{L}_{\pm} = 0$  and  $\mathcal{L}_0$  does not change when  $Z_{\pm}^M$  is changed with  $q_{\mu\nu}$  fixed.

As the second example, let us consider a simple case in which there is only a scalar field other than those matter fields confined on the hypersurface  $\Sigma$ . Namely, let us consider the following Lagrangian densities.

$$\begin{aligned} \mathcal{L}_{\pm} &= -\sqrt{|g|} \left[ \frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi + V_{\pm}(\Phi) \right], \\ \mathcal{L}_0 &= \bar{\mathcal{L}}_0(\phi_{\pm}) + \lambda_{\phi}(\phi_{+} - \phi_{-}), \end{aligned} \quad (27)$$

where  $\bar{\mathcal{L}}_0$  is the Lagrangian density for matter confined on  $\Sigma$ , and  $\phi_{\pm}$  is the pullback of  $\Phi$  on  $\Sigma$  defined by

$$\phi_{\pm}(y) = \Phi|_{x_{\pm}=Z_{\pm}(y)}. \quad (28)$$

The matter Lagrangian density  $\bar{\mathcal{L}}_0$  on  $\Sigma$  can depend on  $\phi_{\pm}$  as well. Note that the Lagrange multiplier field  $\lambda_{\phi}(y)$  is necessary in order that the scalar field should have single value on  $\Sigma$  and that  $\Sigma$  should be regular. For this example we can easily calculate  $T_{\pm}^{MN}$ ,  $S^{\mu\nu}$  and  $F_{\pm M}$  as follows.

$$\begin{aligned} T_{\pm}^{MN} &= \partial_M \Phi \partial_N \Phi - g_{MN} \left[ \frac{1}{2} g^{M'N'} \partial_{M'} \Phi \partial_{N'} \Phi + V(\Phi) \right], \\ S^{\mu\nu} &= \frac{2\epsilon}{\sqrt{|q|}} \frac{\delta}{\delta q_{\mu\nu}(y)} \int_{\Sigma} d^{D-1} y' \bar{\mathcal{L}}_0 \Big|_{\delta\phi_{\pm}=0}. \end{aligned} \quad (29)$$

and

$$F_{+M} = \left[ \frac{1}{2} g^{M'N'} \partial_{M'} \Phi \partial_{N'} \Phi + V_{\pm}(\Phi) \right] n_M + \epsilon(\partial_{\phi_{+}} \bar{\mathcal{L}}_0 + \lambda_{\phi}) \partial_M \Phi, \quad (30)$$

$$F_{-M} = - \left[ \frac{1}{2} g^{M'N'} \partial_{M'} \Phi \partial_{N'} \Phi + V_{\pm}(\Phi) \right] n_M - \epsilon \lambda_{\phi} \partial_M \Phi, \quad (31)$$

where the right hand sides of (30) and (31) are evaluated at  $x_{\pm}^M = Z_{\pm}^M(y)$ , respectively. Hence, by using

$$\lambda_{\phi} = -\partial_{\phi_{+}} \bar{\mathcal{L}}_0 - \epsilon n_{+}^M \partial_M \Phi|_{x_{+}=Z_{+}} = -\epsilon n_{-}^M \partial_M \Phi|_{x_{-}=Z_{-}}, \quad (32)$$

which is a part of equations of motion, it is confirmed that (25) is satisfied. Thus, the consistency condition (25) is actually compatible with equations of motion of the scalar field.

It is easy to extend the above analysis to an arbitrary number of scalar fields.

We have presented an action principle of singular hypersurfaces in general relativity in any dimension without assuming any symmetry. Since an arbitrary number of scalar fields can be consistently included as shown in Sec. IV, the action principle is applicable to a wide class of scalar-tensor type theories of gravity in the Einstein frame. Besides the scalar fields, any kind of matter Lagrangian density on the hypersurface, which may depend also on the pullback of the scalar fields, can be included. The action principle is manifestly doubly covariant in the sense that coordinate systems on and off a hypersurface are disentangled and can be independently specified. More precisely, there are three independent coordinate systems: that on the hypersurface, those in two regions separated by the hypersurface. We have shown that, including variation of the metric, the position of the hypersurface and matter fields, the variational principle gives the correct set of equations of motion: the Einstein equation off the hypersurface, Israel's junction condition in a doubly covariant form and equations of motion of matter fields including the scalar fields. It is worth while mentioning that the position of the hypersurface measured from one side of the hypersurface and that measured from another side can be independently varied as required by the double covariance.

Now let us discuss about application of the doubly covariant action principle to the brane world scenario.

In refs. [23–29] it was shown that the standard cosmology can be realized in the Randall-Sundrum brane world scenario in low energy as far as a spatially homogeneous and isotropic brane is concerned. After that, many authors investigated cosmological perturbations in the brane-world scenario [30,21,31–40].

In particular, four independent equations for scalar perturbations on the brane in the plane symmetric ( $K = 0$ ) background were derived recently by the author in ref. [31]. The number of independent equations is the same as in the standard cosmology, and it was shown that in low energy these sets of equations differ only by the non-local effects due to gravitational waves in the bulk.

In the derivation of the four equations in ref. [31] the author took advantages of the doubly gauge invariant formalism developed in refs. [30,21]. It was essential that the formalism includes perturbation of the position of a brane as a dynamical variable. Actually, as already discussed in ref. [30], if we fix the position of the brane by hand as in the Gaussian normal coordinate system, then it is in general inconsistent with convenient gauge choices in the bulk like a generalized Regge-Wheeler gauge<sup>2</sup>. In other words, as done in refs. [32,30], we can construct  $D$ -gauge invariant variables from the perturbation of the position of the brane, and they are physical degrees of freedom independent of  $D$ -gauge invariant variables in the bulk. The former gauge invariant variables are concise in the sense that it is localized on the brane, and the later variables can be expressed most concisely by the master variables introduced in ref. [26]. Hence, the inclusion of the brane position as a dynamical variable provides us with the most concise configuration space.

Of course, it is always possible to take the Gaussian normal coordinate system. In this coordinate system, degrees of freedom of perturbation of the brane position are included in metric perturbations. Hence, as done in ref. [41] for a static background, we need to extract the information about the brane position from the metric perturbations. Classically, this procedure should not be difficult since we can use equations of motion. However, quantum mechanically, we have to be careful when we use the equations of motion to reduce the action.

The next task in the future is to obtain the second-order action for perturbations by using the doubly covariant action obtained in this paper. After that, we need to obtain the corresponding reduced action by using a formalism to treat constrained systems, eg. Dirac's method [42] or Faddeev-Jackiw method [43]. As shown in ref. [44], perturbative behavior of the Wheeler-de Witt wave function can be investigated by the usual quantum field theory in curved spacetime with the reduced action.

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<sup>2</sup>In the literature it is sometimes called a generalized longitudinal gauge.

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